On the Efficiency of Wage-Setting Mechanisms with Search Frictions and Human Capital Investment

Darong Dai†         Guoqiang Tian‡

This version: November, 2017

Abstract

A challenge facing labor economists is to explain the existence of different wage-setting mechanisms. This paper investigates the relative efficiency of competitive wage, wage bargaining and wage posting. In a search-theoretical model with human capital investment we establish the conditions, which are associated to the cost structure of human capital investment and job vacancy creation, for one of them to prevail. The insight is that a mechanism generates the highest level of equilibrium welfare by achieving the best balance between aggregate output and aggregate cost. We find under the Hosios condition and for a broad range of cost parameter values that: if workers’ matching contribution is sufficiently larger than their output contribution, then wage posting prevails; if their output contribution is sufficiently larger than their matching contribution, then wage bargaining prevails; if the two contributions are sufficiently close to each other, then competitive wage prevails. This finding justifies in some sense the evidences reported by Hall and Krueger (2012) and Brenzel et al. (2014).

Keywords: Competitive wage; Wage bargaining; Wage posting; Welfare comparison; Human capital investment; Search frictions.

JEL classification codes: D40; D61; J63; J64.

1 Introduction

For any equilibrium search theory involving labor market, Rogerson et al. (2005) identify two paramount factors from a huge body of literature: search friction and wage-setting mechanism. Hall and Krueger (2012) provide the survey evidence that wage bargaining and wage posting coexist in the US labor market, and Brenzel et al. (2014) provide the evidence that they coexist in the German labor market. Some

†E-mail: dai496@tamu.edu. Department of Economics, Texas A&M University, College Station 77843, USA.
‡E-mail: gtian@tamu.edu. Department of Economics, Texas A&M University, College Station 77843, USA.
other studies (e.g., Kiyotaki and Moore, 2012; Aruoba et al., 2011), however, highlight the advantage of competitive wage in standard macro models. We hence address the question: which one induces the highest level of social welfare? To our knowledge, our study represents the first attempt to explore optimal wage-setting arrangements in a search theoretical model with human capital investment. We are not motivated to propose a substitutive mechanism but to further enhance our understanding of their relative advantage. In fact, our theory characterizes the links of their relative efficiency with underlying economic environments.

We consider a situation where workers must make human capital investment before finding a job and search frictions prevent ex ante contracting between workers and firms. Thus, workers may underinvest or overinvest, and the entry of firms may be too high or too low, resulting in a too low or too high unemployment rate. In the presence of search frictions, on the one hand, both wages to be received in a match and the job finding probability shape a worker’s incentive of human capital investment; on the other hand, both wages to be paid out and the productivity of the worker to be employed shape a firm’s incentive of job vacancy creation. Their incentives are intertwined through wages, so it is not surprising that we are interested in exploring optimal wage-setting arrangements.

We compare the three wage-setting mechanisms regarding the steady-state social welfare under the assumption of power production function\(^1\) and Cobb-Douglas matching function\(^2\). As the welfare function is nonlinear in three interdependent variables, namely human capital investment, labor-market tightness and search unemployment rate, determined by three nonlinear equations, we can just establish necessary conditions when comparing competitive wage and wage bargaining and also need to impose the restriction that a worker’s marginal contribution rates to production and matching are equal when comparing bargaining and posting. These restrictions must be tolerated for deriving the formal results.

First, competitive wage induces a higher firm entry rate, a lower search unemployment rate and more human capital investments than does wage posting, thereby exhibiting an advantage in aggregate output while a disadvantage in aggregate cost compared to posting. If the marginal cost of job creation is bounded above, then its advantage outweighs its disadvantage, resulting in a higher level of social welfare than under posting. If, however, the marginal cost of job creation is bounded below, then its advantage is dominated by its disadvantage, resulting in a lower level of social welfare than under posting.

Second, suppose Hosios condition\(^3\) holds, then for competitive wage to induce a higher firm entry rate, a lower search unemployment rate and more human capital investments than does wage bargaining, it is necessary that either the worker or the firm has an output share (under competitive wage) greater than the corresponding bargaining share. Thus, if the marginal cost of job creation is bounded above while the average cost of human capital investment is bounded below, to induce a higher level of social

---

\(^1\)Note that when capital input is fixed, Cobb-Douglas or CES production function reduces to power function in labor input.


\(^3\)That is, the elasticity of the matching function with respect to the number of unemployed is equal to the share of workers in the surplus of a match (see Hosios, 1990).
welfare under competitive wage than under bargaining it is necessary that either the worker or the firm has a stronger incentive under competitive wage than under bargaining. If both costs are bounded above, to induce a higher level of social welfare under bargaining than under competitive wage it is necessary that firms’ bargaining share is greater than its output share under competitive wage.

Third, suppose Hosios condition holds and the average cost of human capital investment is bounded below, then posting induces a higher firm entry rate, a lower search unemployment rate and more human capital investments than does bargaining. Thus, if the marginal cost of job creation is bounded above, posting’s relative advantage in aggregate output outweighs its relative disadvantage in aggregate cost, resulting in a higher level of social welfare than under bargaining. If, however, the marginal cost of job creation is bounded below, posting’s relative advantage is dominated by its relative disadvantage, resulting in a lower level of social welfare than under bargaining. These conclusions reveal the relevance of cost structures in mutual welfare comparison.

And fourth, we find under the Hosios condition and for a broad range of cost parameter values that: if workers’ matching contribution (measured by matching elasticity) is sufficiently larger than their output contribution (measured by output elasticity), then wage posting prevails; if their output contribution is sufficiently larger than their matching contribution, then wage bargaining prevails; if the two contributions are sufficiently close to each other, then competitive wage prevails. So, each one of them can be dominant in equilibrium social welfare within a typical region of the two-dimensional parameter space, and a typical wage-setting mechanism should be adopted for labor markets (or jobs) located within the typical region.

Our paper is related to the literature studying the (in)efficiency of (laissez-faire) wage determination in the presence of both human capital investment and labor search frictions. Under competitive wage, Masters (1998) shows that efficiency can be restored, but he assumes a constant wage rate and also that the social planner imposes a specific weight on workers in the social welfare function. Under wage bargaining, many studies (e.g., Acemoglu, 1996; Moen, 1999; Burdett and Smith, 2002; Charlot and Decreuse, 2005, 2010; Charlot et al., 2005; de Meza and Lockwood, 2010) show that efficiency cannot be restored. Under wage posting, Masters (2011) shows that efficiency can be restored whereas Kaas and Zink (2011) show that inefficiency occurs despite competitive search. Although interesting insights have been provided by the literature, there is no consensus on which one should prevail and why. In fact, none of the literature gives an explanation on the coexistence of all three mechanisms in reality.

Departing from the literature, we focus on mutual welfare comparison between any two of the three wage-setting mechanisms, which helps us to figure out which one may prevail in equilibrium and under what conditions. This is technically nontrivial as the equilibrium welfare is a nonlinear function of three interdependent variables determined by three nonlinear equations. Even so, we derive explicit conditions enabling us to predict when one of them is the most efficient. In particular, our theoretical results imply that wage posting does not always dominate the other two and wage bargaining is not necessarily dominated by the other two. Given that the literature has not yet provided a formal theory on why we
should adopt a typical wage-setting mechanism (other than the alternatives), our contribution is to not only rationalize the existence of all three types of wage-setting mechanisms but also identify the links of their relative advantage with the economic environments under consideration.

Though Ellingsen and Rosén (2003) attempt to explain the coexistence of wage bargaining and wage posting, they emphasize the effect of worker heterogeneity and assume that wage policy is unilaterally chosen by firms. In particular, they are interested in circumstances in which wage bargaining is adopted by all firms. Given these differences in economic environments, our study can be interpreted as complementary in terms of explaining the coexistence of alternative wage-setting mechanisms.

The rest of the paper is organized as follows. Section 2 presents the matching framework with some basic assumptions. Section 3 derives decentralized equilibrium under alternative wage-setting mechanisms. Section 4 proceeds to a comparative analysis. Section 5 concludes. Proofs are relegated to Appendix.

2 The Model

2.1 Basic Assumptions

The economy is populated by a continuum of workers and a continuum of firms who are risk-neutral. Workers invest in human capital that is essential for production, while firms create jobs and organize production. The measure of workers is normalized to one. All agents live forever in continuous time and discount the future at the common rate $r > 0$; that is, the rate of time preference is equal to the interest rate under risk neutrality. There is a complete capital market, and all agents have access to this market.

Workers search for a job after they have acquired some human capital $h \geq 0$ at marginal cost $p > 0$, which can be interpreted as the average annual tuition fees. Each firm can create at most one job, and jobs are either filled or vacant. If a firm employs a worker, it produces a flow of output $y = f(h)$ with the price normalized to one, in which $f$ is continuously differentiable, strictly increasing, concave and satisfies Inada conditions. We also let $f(0) = 0$ and $\lim_{h \to 0} hf'(h) = 0$, which are satisfied by power functions such as $f(h) = h^\alpha$ for $\alpha \in (0, 1)$.

The source of job turnover is exogenous and follows a Poisson process with a constant arrival rate $\delta > 0$. Unemployed workers receive a flow payoff of zero. That is, let any unemployment benefits from home production and leisure be normalized to zero. We impose the free entry assumption on firms so they exhaust the rents from job creation in the long run. Additionally, once meetings occur, all payoff relevant characteristics of the other party are revealed so that there is no private information within each match. Here we hold factors such as individual ability and education quality constant and just use the number of years of education to measure the amount of human capital. As it is observable in reality, this assumption regarding information structure is without further loss of generality.
2.2 Random Matching

Unemployed workers and firms with vacancies come together in pairs via a matching technology \( M(u, v) \), where \( u \) is the unemployment rate, \( v \) is the measure of vacancies, and \( M \) is concave and homogeneous of degree one in \((u, v)\) with continuous derivatives. This enables us to write the flow rate of match for a vacancy as \( \frac{M(u, v)}{v} \equiv q(\theta) \), where \( q(\theta) < 0 \) and \( \theta \equiv \frac{v}{u} \) is the labor market tightness (or the inverse of queue length). We, as usual, assume that the absolute value of the elasticity of \( q(\theta) \) satisfies \( -\frac{q'(\theta)}{\theta} \frac{\theta}{q(\theta)} = \eta \) for a constant \( \eta \in (0, 1) \). The flow rate of match for an unemployed worker is \( \frac{M(u, v)}{u} \equiv \theta q(\theta) \). In general, \( q(\theta), \theta q(\theta) < \infty \); thus, it takes time for them to find production partners.

We place Inada-type assumptions on \( M \) such that \( \theta q(\theta) \) is strictly increasing in \( \theta \), \( \lim_{\theta \uparrow \infty} q(\theta) = \lim_{\theta \downarrow 0} \theta q(\theta) = 0 \), and \( \lim_{\theta \downarrow 0} q(\theta) = \lim_{\theta \uparrow \infty} \theta q(\theta) = \infty \). Intuitively, when market tightness goes to zero, the arrival rates of trading partners for firms and workers go to infinity and zero, respectively; when \( \theta \) goes to infinity, the opposite holds.

Matches are consummated only if the joint surplus is nonnegative. We need more notations: \( U \) is the value of unemployment, \( W(h) \) is the value of employment for a worker with human capital \( h \), \( V \) is the value of a vacancy, and \( J(h) \) is the value to a firm of filling a job. Therefore, a match is formed only if

\[
W(h) + J(h) \geq U + V, \quad (1)
\]

for any \( h \geq 0 \). In fact, inequality (1) can be interpreted as the group rationality constraint.

2.3 Asset Values

We restrict attention to a steady state. As in Masters (1998), the asset value of unemployment, \( U \), satisfies the Bellman equation:

\[
rU = \max_{h \geq 0} \left\{ -ph + \theta q(\theta) [W(h) - U] \right\}. \quad (2)
\]

Hence the expected income flow when unemployed is equal to the current income, \(-ph\), plus the expected gain from job search, \( \theta q(\theta) [W(h) - U] \). Similarly, the value of employment, \( W(h) \), is defined by the Bellman equation:

\[
rW(h) = w(h) + \delta [U - W(h)], \quad (3)
\]

where \( w(h) \) denotes wages satisfying \( w(0) = 0 \) which is a normalization, \( w'(\cdot) > 0 \) which rationalizes the activity of human capital investment and \( w''(\cdot) \leq 0 \) which can be interpreted as a regularity constraint. To accept a wage offer, individual rationality requires that \( w(h) \geq rU \), in which \( rU \) can be seen as reservation wages.

We next give the value equations of firms. The value of a vacancy satisfies the Bellman equation:

\[
rV = -c + q(\theta) [J(h) - V], \quad (4)
\]

\(^4\)The expected duration of time spent waiting for a match for any agent is the inverse of her arrival rate.
where \( c > 0 \) denotes a constant flow cost of renting a cite. So, the expected income flow associated with a vacancy is equal to the current income flow, \(-c\), plus the expected gain from search, \( q(\theta)[J(h) - V] \). The value to a firm having a vacancy filled is given by:

\[
rf(h) = f(h) - w(h) + \delta[V - J(h)].
\] (5)

To employ a worker with human capital \( h \), individual rationality requires \( f(h) - w(h) \geq -c \). We further assume \( f(h) \geq w(h) \) (i.e., filled vacancies earn non-negative profits) and \( f'(h) > w'(h) \) (i.e., profits are strictly increasing in the human capital input) for \( \forall h > 0 \), in which the former is not restrictive at all while the latter motivates firms to hire workers with higher productivity.

Applying equations (3)-(5) to inequality (1), a match is formed only if

\[
f(h) \geq rU \text{ for } \forall h \geq 0,
\] (6)

where we have used the free entry assumption \( V = 0 \).

### 3 Decentralized Equilibrium Derivation

#### 3.1 Competitive Wage

A steady-state competitive equilibrium must satisfy five conditions: (1) matches are mutually acceptable, so inequality (6) must be satisfied; (2) wages are competitively determined (i.e., agents maximize payoff by taking wages as given and wages are adjusted to clear market); (3) individual rationality constraints facing workers and firms are satisfied; (4) firms creating a job vacancy earn zero profits, so \( V = 0 \); and (5) the flow of workers into and out of unemployment must be equal (i.e., total employment remains constant at the steady state), formally \( \delta(1 - u) = \theta q(\theta)u \).

The following lemma establishes the steady-state equilibrium.

**Lemma 3.1.** For economic environments under consideration, the following statements are true.

(i) The steady-state competitive equilibrium, written as \( \{C, C^*, u^C\} \), satisfies:

\[
f'(h) \left[ \frac{\theta q(\theta)}{r + \delta} \right] = p, \]

\[
c(r + \delta) = [f(h) - w(h)] q(\theta),
\]

and

\[
u = \frac{\delta}{\delta + \theta q(\theta)},
\]

(ii) For \( \Psi(\theta) \equiv \left\{ f \left( (f')^{-1} \left[ \frac{(r+\delta)p}{\theta q(\theta)} \right] \right) - \left[ \frac{(r+\delta)p}{\theta q(\theta)} \right] (f')^{-1} \left[ \frac{(r+\delta)p}{\theta q(\theta)} \right] \right\} q(\theta) \) and \( w(h) = f'(h)h \), if either

\[
\begin{align*}
&f'(h) \frac{h}{f(h)} > \eta, \quad \text{or} \quad f'(h) \frac{h}{f(h)} < \eta, \\
&\lim_{\theta \downarrow 0} \Psi(\theta) < c(r + \delta), \quad \text{or} \quad \lim_{\theta \uparrow \infty} \Psi(\theta) > c(r + \delta), \\
&\lim_{\theta \downarrow 0} \Psi(\theta) > c(r + \delta) \quad \text{or} \quad \lim_{\theta \uparrow \infty} \Psi(\theta) < c(r + \delta)
\end{align*}
\] (10)
holds, then \( \{ h^C, \theta^C, u^C \} \) determined by equations (7)-(9) is unique and also an interior solution.

(iii) If \( q(\theta) = \theta^{-\eta} \) and \( f(h) = h^\alpha \) for a constant \( \alpha \in (0, 1) \), then (10) can be satisfied for \( \alpha \neq \eta \).

Proof. See Appendix.

Equation (7) states that the marginal cost of human capital investment, \( p \), is equal to the expected and discounted marginal productivity, \( \theta q(\theta) f'(h) / (r + \delta) \), under competitive wage. Since each firm can hire at most one worker, equation (8) states that the marginal cost of creating a job, \( c \), is equal to the expected and discounted marginal benefit, \( q(\theta) [f(h) - w(h)] / (r + \delta) \).

### 3.2 Wage Bargaining

A steady-state search equilibrium must satisfy six conditions: (1) matches are mutually acceptable, so inequality (6) must be satisfied; (2) workers make payoff-maximizing investments; (3) wages are determined by bilateral bargaining between matched workers and firms (see Pissarides, 2000, 2009); (4) individual rationality constraints facing workers and firms are satisfied; (5) firms creating a job vacancy earn zero profits, so \( V = 0 \); and (6) the flow of workers into and out of unemployment must be equal.

The following lemma establishes the steady-state equilibrium.

**Lemma 3.2.** For economic environments under consideration, the following statements are true.

(i) The steady-state search equilibrium, written as \( \{ h^B, \theta^B, u^B \} \), satisfies:

\[
f'(h) \left[ \frac{\beta \theta q(\theta)}{r + \delta} \right] = p, \tag{11}
\]

\[
c(r + \delta) = \{(1 - \beta) [f(h) + ph] - \beta c\} q(\theta), \tag{12}
\]

and

\[
u = \frac{\delta}{\delta + \theta q(\theta)}. \tag{13}
\]

(ii) If the following conditions are satisfied:

\[
\begin{align*}
\Phi'(\theta) &< 0, \\
\lim_{\theta \downarrow 0} \Phi(\theta) &> c(r + \delta), \\
\lim_{\theta \uparrow \infty} \Phi(\theta) &< c(r + \delta),
\end{align*}
\]

where \( \Phi(\theta) \equiv (1 - \beta) \left\{ f \left( \left( f' \right)^{-1} \left[ \frac{(r + \delta)p}{\beta \theta q(\theta)} \right] \right) + p(f')^{-1} \left[ \frac{(r + \delta)p}{\beta \theta q(\theta)} \right] \right\} q(\theta) \), then \( \{ h^B, \theta^B, u^B \} \) determined by equations (11)-(13) is unique, and it is also an interior solution.

(iii) If \( q(\theta) = \theta^{-\eta} \) and \( f(h) = h^\alpha \) for a constant \( \alpha \in (0, 1) \), then (14) is satisfied for \( \forall \eta \in \left[ -\frac{1}{1-\alpha}, 1 \right] \).

Proof. See Appendix.
Equation (11) states that the marginal cost of human capital investment, \( p \), is equal to the expected and discounted marginal productivity shared by a worker, \( \beta \theta q(\theta) f'(h) / (r + \delta) \). By plugging equation (11) in (12), we have \( c = \frac{(1 - \beta)q(\theta)}{r + \delta + \beta \theta q(\theta)} \left[ 1 + \frac{hf'(h) \beta q(\theta)}{f(h) r + \delta} \right] f(h) \), which states that the marginal cost of creating a job, \( c \), is equal to the expected and discounted matching surplus shared by a firm.

### 3.3 Wage Posting

Firms commit to and post wage contracts, denoted \((h, w)\), before meeting workers in an effort to attract applicants, while workers can observe all posted wages and then decide which of these to seek. So, firms are market makers who can open submarkets via posting wages, while workers are allowed to adjust their application decisions in response to wage differentials across submarkets (see, e.g., Shimer, 1996; Moen, 1997; Acemoglu and Shimer, 1999). We still use \( \theta \) to denote the inverse of queue length in submarkets. Since matching frictions still exist within each submarket, workers in a submarket offering a wage of \( w \) are hired with probability \( \theta(w)q[\theta(w)] \). In equilibrium, the set of submarkets is complete in the sense that there is no submarket that could be opened that makes some firms and workers better off.

Events proceed as follows: workers first make investments; after observing these investments, each firm posts a wage contract, taking as given the wage contracts of its competitors; then each worker chooses the submarkets to seek, taking as given wage contracts and the search strategies of other workers. A steady-state competitive search equilibrium must satisfy eight conditions: (1) matches within each submarket are mutually acceptable, so inequality (6) must be satisfied for any posted \( w \); (2) workers make payoff-maximizing investments; (3) wage commitments are profit-maximizing; (4) firms creating a job vacancy earn zero profits, so \( V = 0 \); (5) workers direct their search toward the wages that maximize their expected payoff; (6) \( \theta(w) \) is consistent with rational expectations beginning at any decision node (namely sequentially rational); (7) individual rationality constraints facing workers and firms are satisfied; and (8) the flow of workers into and out of unemployment must be equal.

The following lemma establishes the steady-state equilibrium.

**Lemma 3.3.** For economic environments under consideration, the following statements are true.

(i) The steady-state competitive search equilibrium, written as \( \{h^p, \theta^p, u^p\} \), satisfies:

\[
 f'(h) \left[ \frac{\theta q(\theta)}{r + \delta} \right] = p, \quad (15)
\]

\[
 c(r + \delta) = \{ (1 - \eta) [f(h) + ph] - \eta c \theta \} q(\theta), \quad (16)
\]

and

\[
 u = \frac{\delta}{\delta + \theta q(\theta)}. \quad (17)
\]
(ii) If the following conditions are satisfied:

\[
\begin{align*}
\Xi'(\theta) < 0, \\
\lim_{\theta \downarrow 0} \Xi(\theta) > c(r + \delta), \\
\lim_{\theta \uparrow \infty} \Xi(\theta) < c(r + \delta),
\end{align*}
\]

where \( \Xi(\theta) \equiv (1 - \eta) \left\{ f \left( \left( f' \right)^{-1} \left[ \frac{(r+\delta)p}{\theta q(\theta)} \right] \right) + p(f')^{-1} \left[ \frac{(r+\delta)p}{\theta q(\theta)} \right] \right\} q(\theta) \), then \( \{h^P, \theta^P, u^P\} \) determined by equations \((15)-(17)\) is unique, and it is also an interior solution.

(iii) If \( q(\theta) = \theta^{-\eta} \) and \( f(h) = h^\alpha \) for a constant \( \alpha \in (0, 1) \), then \((18)\) is satisfied for \( \forall \eta \in \left[\frac{1}{2-\alpha}, 1\right) \).

Proof. See Appendix.

In equilibrium, all firms offer the same wage and all workers make the same investment. As before, equation \((15)\) states that the marginal cost of human capital investment, \( p \), is equal to the expected and discounted marginal productivity, \( \theta q(\theta) f'(h) / (r + \delta) \). By plugging equation \((15)\) in equation \((16)\), we have

\[
c = \frac{1 - \eta}{r + \delta + \eta \theta q(\theta)} \left[ 1 + \frac{h f'(h) \theta q(\theta)}{f(h)} \right] f(h),
\]

which states that the marginal cost of posting a job, \( c \), is equal to the expected and discounted matching surplus obtained by a firm.

### 4 Steady-State Welfare Comparison

We now compare the three wage-setting mechanisms regarding the steady-state welfare:\(^5\)

\[
\mathcal{W}_j \equiv f \left( h^j \right) \left( 1 - u^j \right) - ph^j u^j - c\theta^j w^j,
\]

for \( \forall j \in \{C, B, P\} \). To assure attractability, we need the following assumption.

**Assumption 4.1.** \( f(h) = h^\alpha \) and \( q(\theta) = \theta^{-\eta} \) for parameters \( \alpha, \eta \in (0, 1) \).

Also, for our interest, we impose another assumption: \(^6\)

**Assumption 4.2.** \( \eta = \beta \in (0, 1) \), i.e., the Hosios condition holds.

First, we get the following four lemmas.

**Lemma 4.1.** Suppose Assumption 4.1 holds and \( \alpha \neq \eta \), then we have \( \theta^C > \theta^P \), \( u^C < u^P \) and \( h^C > h^P \).

Proof. See Appendix.

As already shown in Lemma 3.1, condition \( \alpha \neq \eta \) assures the existence of competitive equilibrium under Assumption 4.1. Lemma 4.1 states that workers make more human capital investments, firm entry rate is higher and search unemployment rate is lower under competitive wage than under wage posting.

---

\(^5\)Under risk-neutral preferences, steady-state welfare is defined as net output of the society.

\(^6\)In fact, we have analyzed the general case without using Assumption 4.2 and find that nothing essentially new arises from relaxing this assumption. Also, Assumption 4.2 is used only when wage bargaining is involved.
Lemma 4.2. Suppose Assumptions 4.1-4.2 hold, then we have: (i) If \(1 - \alpha \geq (1 - \beta)\beta^\alpha/(1 - \alpha)\), then \(\theta^C \neq \theta^B \) and \(u^C \neq u^B\); (ii) For \(\theta^C < \theta^B \) and \(u^C > u^B\), it is necessary that \(\alpha(1 - \beta)\beta^\alpha/(1 - \alpha) < 1 - \alpha < 1 - \beta\) holds; (iii) For \(\theta^C > \theta^B\), \(u^C < u^B\) and \(h^C > h^B\), it is necessary that either \(1 - \alpha < (1 - \beta)\beta^\alpha/(1 - \alpha)\) or \(1 - \alpha > 1 - \beta\) holds; (iv) \(h^C < h^B\) if and only if \(\theta^B / \theta^C > \beta^{1/(\beta - 1)}\).

Proof. See Appendix.

To intuitively interpret Lemma 4.2, we impose a constant wage rate under competitive wage and hence \(\alpha\) denotes the output share of a matched worker while \(1 - \alpha\) for a matched firm. Part (i) gives the sufficient condition under which competitive wage and wage bargaining diverge on market tightness and search unemployment. For more specific comparisons stated in parts (ii)-(iii), we can just establish necessary conditions under the current assumptions. To induce a lower firm entry rate and a higher search unemployment rate under competitive wage than under wage bargaining, part (ii) shows that it is necessary that the firm’s output share under competitive wage is strictly smaller than its bargaining share under wage bargaining. To induce a higher firm entry rate, a lower search unemployment rate and more human capital investments under competitive wage than under wage bargaining, part (iii) shows that it is necessary that either the firm or the worker is provided with a stronger incentive under competitive wage than under wage bargaining. Part (iv) states that a worker makes more human capital investments under wage bargaining than under competitive wage if and only if firm entry rate under the former is sufficiently higher than that under the latter.

Lemma 4.3. Suppose Assumptions 4.1-4.2 hold and \(\alpha = \eta\), then we have:

(i) \(\theta^B \neq \theta^P \) and \(u^B \neq u^P\);

(ii) \(\theta^B > \theta^P\) and \(u^B < u^P\) for \(p < \hat{p}\), and \(\theta^B < \theta^P\), \(u^B > u^P\) and \(h^B < h^P\) for \(p > \hat{p}\), in which

\[
\hat{p} \equiv \left[\frac{\beta^{\hat{p} + \rho(1 - \beta)\beta^{1 - \hat{p}}}}{(\gamma + \delta)\beta^{\hat{p}} - \beta^{\hat{p}} - \beta^{\hat{p}}}ight]^{1/\beta} > 0;
\]

(iii) \(h^B > h^P\) if and only if \(\theta^B / \theta^P > \beta^{1/(\beta - 1)}\).

Proof. See Appendix.

As the equilibrium equations determining market tightness are highly nonlinear under both bargaining and posting, we must rely on the assumption \(\alpha = \eta\) to guarantee attractability. Intuitively, \(\alpha = \eta\) means that a worker’s marginal contribution rates to output and matching are equal. Part (i) shows that bargaining and posting diverge on market tightness and search unemployment. Part (ii) provides more specific predictions: if the average cost of human capital investment is smaller than a threshold, then firm entry rate is higher and search unemployment rate is lower under bargaining than under posting; if the average cost is larger than the threshold, then firm entry rate is lower, search unemployment rate is higher and workers make less human capital investments under bargaining than under posting. Part (iii) shows that workers make more human capital investments under bargaining than under posting if and only if firm entry rate under the former is sufficiently higher than that under the latter.
\[ \frac{\partial W_j}{\partial \theta} = \begin{cases} > 0 & \text{for } c < \Theta, \\ 0 & \text{for } c = \Theta, \\ < 0 & \text{for } c > \Theta, \end{cases} \]

in which
\[ \Theta = \left\{ f'(h(\theta)) - u(\theta) \left[ f'(h(\theta)) + p \right] h'(\theta) - [f(h(\theta)) + ph(\theta)]u'(\theta) \right\} > 0; \]

(ii) For any given \( \theta \in (0, \infty) \) and all \( j \in \{ C, P \} \),
\[ W_j|_{\theta = \hat{\theta}} = \begin{cases} > W_B|_{\theta = \hat{\theta}} & \text{for } p > \hat{p}, \\ = W_B|_{\theta = \hat{\theta}} & \text{for } p = \hat{p}, \\ < W_B|_{\theta = \hat{\theta}} & \text{for } p < \hat{p}, \end{cases} \]

in which \( \hat{p} = \frac{\alpha(\hat{p})^{1/(1-\alpha)} \delta}{(1-\hat{p}^{1/(1-\alpha)})(r+\delta)} > 0. \)

Proof. See Appendix. \( \square \)

Part (i) identifies the conditions under which welfare function is monotone with respect to \( \theta \), enabling us to compare social welfare between competitive wage and wage posting via comparing their \( \theta \)s (see also Figure 17). Controlling for \( \theta \) and identifying a threshold for the average cost of human capital\(^7\) We let \( \alpha = 0.24 \), which is the labor share of the sector including finance, insurance, real estate, rental and leasing. In 2012, this sector’s GDP contribution to the US economy is around 20%, much bigger than any other sectors (see Lawrence, 2015). We let \( \eta = 0.58 \) that is close to 0.54 suggested by Mortensen and Nagypal (2007). Following Pissarides (2009), we set \( r = 0.004 \). And we let \( \delta = 0.036 \) as suggested by Shimer (2012). Finally, we normalize the marginal cost of human capital investment to 1.
investment, part (ii) realizes direct welfare comparison between bargaining and the other two wage-setting mechanisms. Therefore, part (i) combined with part (ii) allows for mutual welfare comparison between the three wage-setting mechanisms.

We now state the welfare-comparison predictions in three propositions.

**Proposition 4.1.** Suppose Assumption 4.1 holds and also \( \alpha \neq \eta \), then we have:

(i) If \( \theta^C \leq \hat{\theta} \), then \( W^C > W^P \);

(ii) If \( \theta^P \geq \hat{\theta} \), then \( W^C > W^P \) for \( c < \Theta \) while \( W^C < W^P \) for \( c > \Theta \).

*Proof.* An application of Lemmas 4.4 and 4.1.

Compared to wage posting, Lemma 4.1 confirms the relative advantage of competitive wage in search unemployment and aggregate output as well as its relative disadvantage in the cost of human capital investment and job creation. Proposition 4.1 shows that the relative advantage dominates the relative disadvantage when the cost of creating a job vacancy is bounded above, otherwise the relative advantage is dominated by the relative disadvantage when the cost of creating a job vacancy is bounded below.

**Proposition 4.2.** Suppose Assumptions 4.1-4.2 hold, then we have:

(i) If \( p \geq \hat{p} \) and \( c < \Theta \), then for \( W^C > W^B \) it is necessary that either \( 1 - \alpha < (1 - \beta)\beta^\alpha/(1 - \alpha) \) or \( 1 - \alpha > 1 - \beta \) holds;

(ii) If \( p \geq \hat{p} \) and \( c > \Theta \), then for \( W^C > W^B \) it is necessary that \( \alpha(1 - \beta)\beta^\alpha/(1 - \alpha) < 1 - \alpha < 1 - \beta \) holds;

(iii) If \( p \leq \hat{p} \) and \( c > \Theta \), then for \( W^C < W^B \) it is necessary that either \( 1 - \alpha < (1 - \beta)\beta^\alpha/(1 - \alpha) \) or \( 1 - \alpha > 1 - \beta \) holds;

(iv) If \( p \leq \hat{p} \) and \( c < \Theta \), then for \( W^C < W^B \) it is necessary that \( \alpha(1 - \beta)\beta^\alpha/(1 - \alpha) < 1 - \alpha < 1 - \beta \) holds.

*Proof.* An application of Lemmas 4.4 and 4.2.

We can just establish necessary conditions based on Lemma 4.2. If the average cost of human capital investment and the marginal cost of job creation are bounded, either from above or below, to establish a welfare ranking between competitive wage and wage bargaining it is necessary that either the output share of a firm under competitive wage is smaller than its bargaining share or at least one side of a matched worker-firm pair has an output share greater than the corresponding bargaining share.

**Proposition 4.3.** Suppose Assumptions 4.1-4.2 hold and also \( \alpha = \eta \), then we have:

(i) If either \( c < \Theta \) and \( p > \max \{ \hat{p}, \tilde{p} \} \) or \( \Theta < c < \Lambda \) and \( \tilde{p} \leq p < \hat{p} \) hold, then \( W^P > W^B \);

(ii) If either \( c < \Theta \) and \( p < \min \{ \hat{p}, \tilde{p} \} \) or \( c > \max \{ \Theta, \Lambda \} \) and \( \hat{p} < p \leq \tilde{p} \) hold, then \( W^P < W^B \), in which \( \Lambda \equiv \left[ \frac{\beta(1 - \beta)^{-\alpha} - (1 - \beta)\beta^{-\alpha} + \alpha^\beta}{(1 - \beta)^{(1 - \alpha)}} \right]^{1/(1 - \beta)} > 0 \).
\[
\eta = \frac{1}{2 - \alpha}
\]

Figure 2: Relative efficiency: \( \eta = \beta, r + \delta = 1, p = 1 \) (normalization) and \( c > 1 \).

**Proof.** An application of Lemmas 4.4 and 4.3.

Proposition 4.3 provides sufficient conditions consisting of restrictions placed on the average cost of human capital investment and the marginal cost of job creation such that either posting strictly dominates bargaining or bargaining strictly dominates posting. First, even if bargaining induces a higher firm entry rate and a lower search unemployment rate than does posting, it may still be dominated by posting in welfare. Second, even if posting induces a higher firm entry rate, a lower search unemployment rate and more human capital investments, it may still be dominated by bargaining in welfare. Therefore, the cost structures of human capital investment and job vacancy creation are relevant factors for the current comparative analysis with respect to steady-state welfare.

Moreover, as roughly illustrated by Figure 2\(^8\), we can find at least one region of the parameter space generated by \( \alpha \) and \( \eta \) such that a typical mechanism prevails within the region. For the specified parameter values, we obtain the following nontrivial observations. If workers’ matching contribution measured by \( \eta \) is sufficiently larger than their output contribution measured by \( \alpha \), then wage posting (WP) prevails. If, in contrast, their output contribution is sufficiently larger than their matching contribution, then wage bargaining (WB) prevails. If their output and matching contributions are sufficiently close to each other, then competitive wage (CW) prevails. Our findings reveal that the three wage-setting mechanisms do exhibit different relative advantages and also can be interpreted as being consistent with the evidences reported by Hall and Krueger (2012) and Brenzel et al. (2014): bargaining is common by minority workers.

\(^8\)As a caveat, we leave some blank areas and use dotted lines because it is unlikely to explicitly identify the partitioning boundaries. Note that the welfare function is highly nonlinear and endogenous variables cannot be explicitly derived under bargaining and posting, we can just rely on numerical simulations to search for the region within which a mechanism prevails.
with professional degrees while posting is particularly common among union members, those who took
government jobs and non-high-school graduates.

5 Conclusion

This paper evaluates the allocative performance of three wage-setting mechanisms with human capital
investment and search frictions. We identify explicit conditions under which one mechanism performs
better than the others in generating a higher level of equilibrium social welfare. Our results reveal that
none of them can unconditionally dominate the other two. To balance between aggregate output and
aggregate cost, the cost structures of human capital investment and job vacancy creation turn out to be
relevant. There exist reasonable ranges of cost parameters such that each one of the three mechanisms
can generate the highest level of equilibrium social welfare. In particular, even though workers tend to
be held up and hence underinvest under bargaining, we find that it may still dominate the other two
in equilibrium welfare. The intuition is that its relative advantage in saving aggregate cost via inducing
low investments and a small number of job openings may outweigh its relative disadvantage in aggregate
output, hence dominating the other two in net output of the society. Our contribution is thus to identify
the associated boundaries under the current context of economy.

These results show that different wage-setting mechanisms have different implications for the nature
of equilibrium. They help to sort out when the inefficiency arising from search-theoretical models of labor
market is due to features of the environment (such as preferences and information) and when it is due
to the assumed wage-setting mechanism, which is especially informative for policymakers. Moreover, as
our results are derived based on a laissez-faire economy, they help to identify along which dimension
and to what extent government intervention should be. For example, we conjecture that, ceteris paribus,
schooling subsidies, which lift up the threat point of workers and alleviate the tension of holdups, should
be more desirable under bargaining while laws guaranteeing the enforcement of labor contracts is more
desirable under posting.

---

9In fact, Flinn and Mullins (2015) find that policies like schooling subsidies attempting to redistribute the surplus between
firms and workers tend to promote more schooling investments and lead to efficiency gains.
References


Appendix: Proofs

Proof of Lemma 3.1: We shall complete it in 5 steps.

Step 1. By equation (3), we have \( W(h^*) = \frac{w(h^*)}{(r + \delta)} \), in which \( h^* \) denotes the supply of human capital. Plugging it in equation (2) and rearranging the algebra, payoff-maximizing investment/supply of human capital implies that \((r + \delta)p = \theta q(\theta)w'(h^*)\). Correspondingly, we can rewrite equation (5) as \((r + \delta)f(h^d) = f(h^d) - w(h^d)\), where \( h^d \) denotes the demand of human capital and we have used the free entry assumption \( V = 0 \). Thus, maximizing \( J(h^d) \) yields that \( f'(h^d) = w'(h^d) \). Competitive equilibrium requires that \( h^d = h^* \), and also it is determined by equation \((r + \delta)p = \theta q(\theta)f'(h)\).

Step 2. Since we have \( J(h) = [f(h) - w(h)] / (r + \delta) \), applying \( V = 0 \) to equation (4) produces that \( c(r + \delta) = [f(h) - w(h)]q(\theta) \), which determines the equilibrium market tightness \( \theta \). In the steady state, the flow of workers into unemployment, \( \delta(1 - u) \), must be equal to the flow of workers out of unemployment, \( \theta q(\theta)u \). To conclude, the steady-state equilibrium must simultaneously satisfy (7)-(9). Additionally, reservation wages evaluated at the equilibrium can be written as \( rU = \frac{\theta q(\theta)}{r + \delta + \theta q(\theta)} [w(h) - f'(h)h] \), which implies that \( w(h) \geq rU \). That is, individual rationality constraint is fulfilled for both workers and firms.

Step 3. We now consider a case where firms offer a constant wage rate, then equilibrium wages can be written as \( w(h) = f'(h)h \). Applying Implicit Function Theorem and differentiating both sides of \((r + \delta)p = \theta q(\theta)f'(h)\) with respect to \( \theta \), we have

\[
-\theta f''(h) \frac{\partial h}{\partial \theta} = (1 - \eta) f'(h),
\]

which implies that \( \partial h / \partial \theta > 0 \). We can hence rewrite \( c(r + \delta) = [f(h) - f'(h)h]q(\theta) \) as

\[
c(r + \delta) = \left\{ \frac{f [h(\theta)] - f' [h(\theta)]}{h(\theta)} \right\} q(\theta),
\]

where \( h(\theta) = (f')^{-1} \left[ \frac{(r+\delta)p}{\theta q(\theta)} \right] \). Noting that \( \Psi(\theta) = \frac{\partial(\theta)}{\partial q(\theta)} f'[h(\theta)]h(\theta) + \left( f'[h(\theta)] - f'(h) \right) q(\theta) \), where we have used equation (19), then we can get that

\[
\Psi'(\theta) > 0 \iff f'[h(\theta)] \frac{h(\theta)}{f'[h(\theta)h]} > \eta \quad \text{and} \quad \Psi'(\theta) < 0 \iff f'[h(\theta)] \frac{h(\theta)}{f'[h(\theta)h]} < \eta.
\]

As a consequence, if condition (10) holds, we have a unique solution of \( \theta \) that is also an interior solution. It is easy to verify that both equilibrium \( h \) and equilibrium \( u \) are uniquely determined, and they are interior solutions as well.

Step 4. We show that condition (10) can be satisfied for reasonable functional forms of \( f \) and \( q \). Let \( q(\theta) = \theta^{-\eta} \) and \( f(h) = h^\alpha \) for a constant \( \alpha \in (0, 1) \), then we obtain \( \Psi(\theta) = (1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right] ^{\frac{1}{\alpha}} \theta^{\alpha(1-\eta)} - \eta \).

Therefore, as long as \( \alpha \neq \eta \), equilibrium \( \theta \) is uniquely determined, and it is an interior solution.

Step 5. We finally verify that the matches with competitive wage are in the mutual interest of workers and firms. In the equilibrium, we have

\[
f(h) - rU = \frac{(r + \delta)f(h) + \theta q(\theta) [f(h) - w(h) + f'(h)h]}{r + \delta + \theta q(\theta)},
\]
it is thus immediate that \( f(h) \geq w(h) \Rightarrow f(h) \geq rU \) for \( \forall h \geq 0 \). By using our assumption, group rationality constraint is satisfied in the competitive equilibrium. Q.E.D.

**Proof of Lemma 3.2:** We shall complete it in 5 steps.

**Step 1.** Since (ex ante) human capital investment and (ex post) wage bargaining move sequentially in this case, we apply backward induction to derive the search equilibrium. We first derive the Nash wage. The generalized Nash bargaining solution selects a wage contract \( w(h) \) to maximize \( [W(h) - U]^{\beta} [J(h) - V]^{1-\beta} \) for some given \( \beta \in (0, 1) \). Performing this maximization yields a necessary and sufficient first-order condition:

\[
(1 - \beta) [W(h) - U] = \beta [J(h) - V].
\]

So, the worker receives a fraction \( \beta \) of total match surplus: \( W(h) - U = \beta [W(h) + J(h) - U - V] \). Applying equations (3) and (5) to equation (20) produces that

\[
w(h) = \beta f(h) + (1 - \beta)rU.
\]

The Nash wage is thus a weighted average of the output of the match and the worker’s reservation wages.

**Step 2.** We now establish the three equations determining the steady-state equilibrium. First, making use of equations (2), (3) and (21), we get that

\[
rU = \max_{h \geq 0} \left\{ \frac{-(r + \delta)ph + \beta\theta q(\theta)f(h)}{r + \delta + \beta\theta q(\theta)} \right\}.
\]

Thus, payoff maximization yields a necessary and sufficient first-order condition (11), which determines the equilibrium level of human capital investment. Second, making use of \( V = 0 \) and equations (4), (5), (21) and (22), we get (12), which determines the equilibrium market tightness. The flow of workers into unemployment, \( \delta(1 - u) \), must be equal to the flow of workers out of unemployment, \( \theta q(\theta)u \), we thus have equation (13).

We next verify that individual rationality constraints facing workers and firms are satisfied. For workers, evaluating equation (22) at the equilibrium outcome and substituting it into equation (21) and simplifying the algebra, we then get the equilibrium Nash wage contract as

\[
w(h) = \beta \left\{ \frac{(r + \delta)f(h) + \theta q(\theta)[f(h) - f'(h)h] + \beta\theta q(\theta)f'(h)h}{r + \delta + \beta\theta q(\theta)} \right\} \geq 0
\]

for \( \forall h \geq 0 \) and \( \forall \theta \geq 0 \), where we have used the equilibrium equation (11) as well as the canonical assumptions placed on \( f \). For firms, equations (22) and (21) imply that in equilibrium:

\[
f(h) - w(h) = \frac{(1 - \beta)(r + \delta)[f(h) + ph]}{r + \delta + \beta\theta q(\theta)} \geq 0
\]

for \( \forall h \geq 0 \). We, accordingly, claim that the individual rationality constraints facing workers and firms are fulfilled under wage bargaining.
Step 3. Here we provide a group of sufficient conditions guaranteeing that we have a unique equilibrium that is also an interior equilibrium. First, by exploiting equation (11) and the strict concavity of \( f \), we get \( h(\theta) = (f')^{-1} \left[ \frac{(r + \delta)p}{\beta \theta q(\theta)} \right] \). Plugging this \( h(\theta) \) in (12) and rearranging the algebra, we have

\[
\frac{c(r + \delta) + \beta c \theta q(\theta)}{\Phi(\theta)} = (1 - \beta) \left\{ f \left( (f')^{-1} \left[ \frac{(r + \delta)p}{\beta \theta q(\theta)} \right] \right) + p(f')^{-1} \left[ \frac{(r + \delta)p}{\beta \theta q(\theta)} \right] \} q(\theta). \tag{23}
\]

By our assumptions, we have \( \phi'(\theta) > 0 \), \( \lim_{\beta \downarrow 0} \phi(\theta) = c(r + \delta) \) and \( \lim_{\beta \uparrow \infty} \phi(\theta) = \infty \). Therefore, to make equation (23) have a unique solution of \( \theta \) that is also an interior solution, we can impose restrictions on \( f \) and \( q \) such that \( \Phi(\theta) \) satisfies: \( \Phi'(\theta) < 0 \), \( \lim_{\beta \downarrow 0} \Phi(\theta) > c(r + \delta) \) and \( \lim_{\beta \uparrow \infty} \Phi(\theta) < c(r + \delta) \).

Step 4. We now confirm that we can find reasonable functional forms of \( f \) and \( q \) so that the sufficient conditions derived in Step 3 are satisfied. Let \( q(\theta) = \theta^{-\eta} \) and \( f(h) = h^a \) for a constant \( a \in (0, 1) \), then we have

\[
\Phi(\theta) = (1 - \beta) \left\{ \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{a - \eta}} \theta^{\frac{a(1 - \eta)}{a - \eta}} + p \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{a - \eta}} \theta^{\frac{1 - \eta}{a - \eta}} \right\}.
\]

If \( \eta \geq \frac{1}{a - \eta} \), then \( \Phi(\theta) \) satisfies the requirements proposed in Step 3. That is, equation (23) indeed determines a unique \( \theta \) that belongs to \((0, \infty)\).

Step 5. We now verify that the matches with wage bargaining are in the mutual interest of workers and firms. By equation (22), we have in equilibrium: \( f(h) \geq rU \Leftrightarrow (r + \delta)[f(h) + ph] \geq 0 \), which thus shows that inequality (6) always holds true. Q.E.D.

**Proof of Lemma 3.3:** We shall complete it in 4 steps.

Step 1. Using equations (2) and (3), we have

\[
rU = \max_{h \geq 0} \left\{ \frac{-(r + \delta)ph + \theta q(\theta)w(h)}{r + \delta + \theta q(\theta)} \right\}. \tag{24}
\]

Using equations (4) and (5), we have

\[
rV = \frac{-(r + \delta) + \theta q(\theta)[f(h) - w(h)]]}{r + \delta + q(\theta)}.
\]

Similar to Lemma 1 of Acemoglu and Shimer (1999), we can characterize the steady-state equilibrium under wage posting as a solution to the constrained maximization problem:

\[
\max_{h, w, \theta} \frac{-(r + \delta)ph + \theta q(\theta)w}{r + \delta + \theta q(\theta)}
\]

subject to \( rV \geq 0 \), i.e., \( q(\theta)[f(h) - w] \geq c(r + \delta) \). That is, competitive search equilibrium should select the posted wage contract \((h, w)\) and the market tightness that maximize workers’ payoff and simultaneously assure that firms are willing to create job vacancies (i.e., the profits earning from job vacancy creation should be non-negative). The Lagrangian can be written as

\[
\mathcal{L}(h, w, \theta; \mu) = \frac{-(r + \delta)ph + \theta q(\theta)w}{r + \delta + \theta q(\theta)} + \mu \left\{ q(\theta)[f(h) - w] - c(r + \delta) \right\},
\]
for a Lagrangian multiplier $\mu \geq 0$. We hence obtain a group of necessary first-order conditions:

$$\frac{\partial L}{\partial h} = \frac{-(r + \delta)p}{r + \delta + \theta q(\theta)} + \mu q(\theta)f'(h) = 0,$$

$$\frac{\partial L}{\partial w} = \frac{\theta q(\theta)}{r + \delta + \theta q(\theta)} - \mu q(\theta) = 0,$$

$$\frac{\partial L}{\partial \theta} = \frac{(1 - \eta)q(\theta) w}{r + \delta + \theta q(\theta)} - \frac{(1 - \eta) [-(r + \delta)ph + \theta q(\theta)w] q(\theta)}{[r + \delta + \theta q(\theta)]^2} + \mu [f(h) - w]q'(\theta) = 0.$$  

(25)

(26)

(27)

It follows from equation (25) or equation (26) that $\mu > 0$, and hence

$$q(\theta)[f(h) - w] = c(r + \delta)$$  

(28)

by using the complementary slackness condition. Equations (25)-(26) imply equilibrium equation (15). Using equations (26) and (28), we can simplify equation (27) and obtain equilibrium equation (16). In addition, the flow of workers into unemployment, $\delta(1-u)$, must be equal to the flow of workers out of unemployment, $\theta q(\theta)u$, we hence have equilibrium equation (17).

In addition, we need to verify whether or not the individual rationality constraints are satisfied in equilibrium. Equation (28) immediately implies that $f(h) - w \geq 0$ for $\forall h \geq 0$ and $\theta \geq 0$. So, the individual rationality constraint facing firms is satisfied. Using equations (28) and (16), we have $w + ph = \eta[f(h) + ph + c\theta] \geq 0$ for $\forall h \geq 0$ and $\theta \geq 0$. So, the individual rationality constraint facing workers is satisfied.

Step 2. Here we provide a group of sufficient conditions guaranteeing that we have a unique equilibrium that is also an interior equilibrium. First, by exploiting equation (15) and the strict concavity of $f$, we get $h(\theta) = (f')^{-1}\left[\frac{(r + \delta)p}{\theta q(\theta)}\right]$. Then, plugging this $h(\theta)$ in (16) and rearranging the algebra, we have

$$\frac{c(r + \delta) + \eta \theta q(\theta)}{\Xi(\theta)} = (1 - \eta) \left\{ f\left( (f')^{-1}\left[\frac{(r + \delta)p}{\theta q(\theta)}\right] \right) \right\} + p(f')^{-1}\left[\frac{(r + \delta)p}{\theta q(\theta)}\right] q(\theta).$$

(29)

By our assumptions, we have $\chi'(\theta) > 0$, $\lim_{\theta \downarrow \alpha} \chi(\theta) = c(r + \delta)$ and $\lim_{\theta \uparrow \infty} \chi(\theta) = \infty$. Therefore, to make equation (29) have a unique solution of $\theta$ that is also an interior solution, we can impose restrictions on $f$ and $q$ such that $\Xi(\theta)$ satisfies: $\Xi'(\theta) < 0$, $\lim_{\theta \downarrow \alpha} \Xi(\theta) > c(r + \delta)$ and $\lim_{\theta \uparrow \infty} \Xi(\theta) < c(r + \delta)$.

Step 3. We now confirm that we can find reasonable functional forms of $f$ and $q$ so that the sufficient conditions derived in Step 2 are satisfied. Let $q(\theta) = \theta^{-\eta}$ and $f(h) = h^\alpha$ for a constant $\alpha \in (0,1)$, then we have

$$\Xi(\theta) = (1 - \eta) \left\{ \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1-\eta}} \theta^{\frac{(1 - \eta)}{1 - \alpha}} - \eta + p \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1-\eta}} \theta^{\frac{1 - \eta}{1 - \alpha} - \eta} \right\}. $$

If $\eta \geq \frac{1}{2 - \alpha}$, then $\Xi(\theta)$ satisfies the requirements proposed in Step 2. That is, equation (29) indeed determines a unique $\theta$ that belongs to $(0,\infty)$.

Step 4. We now verify that the competitive search equilibrium fulfills that matches with wage posting are in the mutual interest of firms and workers. Note that $f(h) \geq rU \iff f(h) + ph + c\theta \geq 0$ by applying
Proof of Lemma 4.1: We shall complete it in 4 steps.

Step 1. Applying Assumption 4.1 to (7)-(8) produces:

\[
(1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} \left( \theta^c \right)^{\frac{\alpha}{r-1}} = c(r + \delta) \iff \theta^c = \left\{ \left[ \frac{1 - \alpha}{c(r + \delta)} \right] \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} \right\}^{\frac{1}{1-p}} \]  

(30)

for \( \alpha \neq \eta \). Similarly, applying Assumption 4.1 to (15)-(16) produces:

\[
(1 - \eta) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} + (1 - \eta) \left[ \frac{\eta}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} \left( \theta^p \right)^{\frac{\alpha}{r-1}} - \eta c \left( \theta^p \right)^{1-\eta} = c(r + \delta). \]  

(31)

Step 2. We first prove that \( \theta^c \neq \theta^p \), and we prove it by means of contradiction. Suppose \( \theta^c = \theta^p \), then we get from (30)-(31) that

\[
\theta^p < \left\{ \left( \frac{1 - \eta}{\eta} \right) \left( \frac{p}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} \right\}^{\frac{1-p}{1-p}}, \]  

(32)

for either \( \alpha > \eta \) or \( \alpha < \eta \). If \( \alpha > \eta \), then combining (30) with (32) shows that

\[
\left[ \frac{1 - \alpha}{c(r + \delta)} \right] \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} > \left( \frac{1 - \eta}{\eta} \right) \left( \frac{p}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} \iff \alpha < \eta,
\]

a contradiction. If \( \alpha < \eta \), then combining (30) with (32) shows that

\[
\left[ \frac{1 - \alpha}{c(r + \delta)} \right] \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} < \left( \frac{1 - \eta}{\eta} \right) \left( \frac{p}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} \iff \alpha > \eta,
\]

a contradiction. Thus, we should have \( \theta^c \neq \theta^p \).

Step 3. If we assume in (30) and (31) that

\[
(1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} \left( \theta^c \right)^{\frac{\alpha}{r-1}} = (1 - \eta) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} \left( \theta^p \right)^{\frac{\alpha}{r-1}},
\]

then we have

\[
\frac{\theta^p}{\theta^c} = \left( \frac{1 - \alpha}{1 - \eta} \right)^{\frac{1}{1-p}} \]  

(33)

as well as

\[
\theta^p = \left\{ \left( \frac{1 - \eta}{\eta} \right) \left( \frac{p}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right] \right\}^{\frac{1}{1-p}} \]  

(34)

by using (30) and (31) again. Combining (30) with (34) shows that

\[
\frac{\theta^p}{\theta^c} = \left[ \frac{(1 - \eta)\alpha}{\eta(1 - \alpha)} \right]^{\frac{1}{1-p}},
\]

plugging which in (33) results in \( \alpha = \eta \), a contradiction. Thus, we should have

\[
(1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} \left( \theta^c \right)^{\frac{\alpha}{r-1}} \neq (1 - \eta) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{1-p}} \left( \theta^p \right)^{\frac{\alpha}{r-1}}.
\]
Step 4. We then prove that $\theta^c < \theta^p$ cannot hold, and we prove it by means of contradiction. Suppose, instead, that $\theta^c < \theta^p$ holds true. We need to consider two cases. First, suppose

$$
(1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\beta}} \left( \theta^c \right)^{\frac{\alpha - \eta}{1 - \eta}} > (1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\beta}} \left( \theta^p \right)^{\frac{\alpha - \eta}{1 - \eta}},
$$

then we have $\left( \frac{\theta^p}{\theta^c} \right)^{\frac{\alpha - \eta}{1 - \eta}} < \frac{1 - \alpha}{1 - \eta}$. If $\alpha > \eta$, then $\frac{\theta^p}{\theta^c} < \left( \frac{1 - \alpha}{1 - \eta} \right)^{\frac{1 - \eta}{\alpha - \eta}} < 1$, which however contradicts with the assumption that $\theta^c < \theta^p$. So, we assume that $\alpha < \eta$. Also, applying (35) to (30)-(31) shows that (32) is satisfied. Thus $\theta^c < \theta^p$ combined with (30) yields that

$$
\left[ \frac{1 - \alpha}{c(r + \delta)} \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\beta}} \left( \theta^c \right)^{\frac{\alpha - \eta}{1 - \eta}} < \left( \frac{1 - \eta}{\eta} \right) \left( \frac{p}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\beta}} \Leftrightarrow \alpha > \eta,
$$
a contradiction. Thus, we consider the second case:

$$
(1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\beta}} \left( \theta^c \right)^{\frac{\alpha - \eta}{1 - \eta}} < (1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\beta}} \left( \theta^p \right)^{\frac{\alpha - \eta}{1 - \eta}},
$$

which implies $\left( \frac{\theta^p}{\theta^c} \right)^{\frac{\alpha - \eta}{1 - \eta}} > \frac{1 - \alpha}{1 - \eta}$. If $\alpha < \eta$, then $\frac{\theta^p}{\theta^c} > \left( \frac{1 - \alpha}{1 - \eta} \right)^{\frac{1 - \eta}{\alpha - \eta}} > 1$, which however contradicts with the assumption that $\theta^c < \theta^p$. So, we assume that $\alpha > \eta$. Also, applying (36) to (30)-(31) shows that (32) is satisfied. Thus $\theta^c < \theta^p$ combined with (30) yields that

$$
\left[ \frac{1 - \alpha}{c(r + \delta)} \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\beta}} \left( \theta^c \right)^{\frac{\alpha - \eta}{1 - \eta}} > \left( \frac{1 - \eta}{\eta} \right) \left( \frac{p}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\beta}} \Leftrightarrow \alpha < \eta,
$$
a contradiction. Thus we should have $\theta^c > \theta^p$ other than $\theta^c < \theta^p$. Finally, note that $h^i = \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\beta}} \left( \theta^i \right)^{\frac{1 - \eta}{\alpha - \eta}}$ and $u^i = \delta / \left[ \delta + (\theta^i)_{1-\eta} \right]$ for $\forall j \in \{C, P\}$ under Assumption 4.1, thus the proof is complete. Q.E.D.

**Proof of Lemma 4.2:** We shall complete it in 6 steps.

**Step 1.** Applying Assumption 4.1 to (11)-(12) produces:

$$
(1 - \beta) \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\beta}} \left( \theta^B \right)^{\frac{\alpha - \eta}{1 - \eta}} + (1 - \beta) p \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\beta}} \left( \theta^B \right)^{\frac{\alpha - \eta}{1 - \eta}} - \beta c \left( \theta^B \right)^{1-\eta} = c(r + \delta),
$$

which combined with (30) shows that

$$
(1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\beta}} \left( \theta^C \right)^{\frac{\alpha - \eta}{1 - \eta}} = (1 - \beta) \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\beta}} \left( \theta^B \right)^{\frac{\alpha - \eta}{1 - \eta}}
$$

+ $(1 - \beta) p \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\beta}} \left( \theta^B \right)^{\frac{\alpha - \eta}{1 - \eta}} - \beta c \left( \theta^B \right)^{1-\eta}. \quad (37)$

**Step 2.** We prove part (i) by means of contradiction, namely we assume that $\theta^c = \theta^B$. First, if $1 - \alpha = (1 - \beta) \beta^{\frac{1}{\beta}}$, then it follows from (37) that

$$
\theta^B = \left\{ \left( 1 - \beta \right) \beta^{\frac{1}{\beta}} \left( \frac{p}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right] \right\}^{\frac{1}{\beta}}. \quad (38)
$$
So, $\theta^C = \theta^B$ implies that
\[
\left[ \frac{1 - \alpha}{c(r + \delta)} \right] \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \eta}} = (1 - \beta)\beta^{\frac{1}{1 - \eta}} \left( \frac{p}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \eta}} \iff 1 - \alpha = \alpha(1 - \beta)\beta^{\frac{1}{1 - \eta}},
\]
where we have used (30). But it contradicts with the assumption $1 - \alpha = (1 - \beta)\beta^{\frac{1}{1 - \eta}}$. Second, if we assume $1 - \alpha > (1 - \beta)\beta^{\frac{1}{1 - \eta}}$, then we get from (37) that
\[
\left( \theta^B \right)^{\frac{1}{1 - \eta}} < (1 - \beta)\beta^{\frac{1}{1 - \eta}} \left( \frac{p}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \eta}},
\]
(39)
Next, if $\alpha > \eta$, then (39) combined with $\theta^C = \theta^B$ and (30) yields
\[
\left[ \frac{1 - \alpha}{c(r + \delta)} \right] \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \eta}} < (1 - \beta)\beta^{\frac{1}{1 - \eta}} \left( \frac{p}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \eta}} \iff 1 - \alpha < \alpha(1 - \beta)\beta^{\frac{1}{1 - \eta}} < (1 - \beta)\beta^{\frac{1}{1 - \eta}},
\]
a contradiction. If $\alpha < \eta$, then we can show that (39) combined with $\theta^C = \theta^B$ and (30) yields the same contradiction. Therefore, the required assertion in part (i) follows.

Step 3. We now show that
\[
(1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \eta}} \left( \theta^C \right)^{\frac{1}{1 - \eta}} \neq (1 - \beta) \left[ \frac{\alpha\beta}{(r + \delta)p} \right]^{\frac{1}{1 - \eta}} \left( \theta^B \right)^{\frac{1}{1 - \eta}},
\]
and we prove this by means of contradiction. Suppose, instead, that
\[
(1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \eta}} \left( \theta^C \right)^{\frac{1}{1 - \eta}} = (1 - \beta) \left[ \frac{\alpha\beta}{(r + \delta)p} \right]^{\frac{1}{1 - \eta}} \left( \theta^B \right)^{\frac{1}{1 - \eta}},
\]
(40)
which yields
\[
\frac{\theta^B}{\theta^C} = \left[ \frac{1 - \alpha}{1 - \beta} \right]^{\frac{1}{1 - \eta}} \beta^{\frac{1}{1 - \eta}}.
\]
(41)
Applying (40) to (37) gives rise to (38), which combined with (30) shows that
\[
\frac{\theta^B}{\theta^C} = \left[ \frac{1 - \alpha}{\alpha(1 - \beta)\beta^{\frac{1}{1 - \eta}}} \right]^{\frac{1}{1 - \eta}},
\]
which combined with (41) implies that $\alpha = 1$, a contradiction.

Step 4. We now show that
\[
(1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \eta}} \left( \theta^C \right)^{\frac{1}{1 - \eta}} > (1 - \beta) \left[ \frac{\alpha\beta}{(r + \delta)p} \right]^{\frac{1}{1 - \eta}} \left( \theta^B \right)^{\frac{1}{1 - \eta}}
\]
does not hold. We also prove this by means of contradiction. Suppose it does hold, then we have $(\theta^B / \theta^C)^{\frac{1}{1 - \eta}} < (1 - \alpha) / [(1 - \beta)\beta^{\frac{1}{1 - \eta}}]$. If $\alpha > \eta$, then we have
\[
\theta^B < \left[ \frac{1 - \alpha}{(1 - \beta)\beta^{\frac{1}{1 - \eta}}} \right]^{\frac{1}{1 - \eta}} \theta^C.
\]
(42)
Also, note from (37) that in this case we can have

$$\theta^B > \left\{ (1 - \beta) \beta^{\frac{\alpha}{r}} \left( \frac{P}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \alpha}} \right\}^{\frac{1}{\frac{1}{1 - \alpha}}}. \quad (43)$$

Making use of (30), (42) and (43), we obtain

$$\left[ \frac{1 - \alpha}{(1 - \beta)\beta^{\frac{\alpha}{r}}} \right]^{\frac{1}{\frac{1}{1 - \alpha}}} \theta^C > \left\{ (1 - \beta) \beta^{\frac{\alpha}{r}} \left( \frac{P}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \alpha}} \right\}^{\frac{1}{\frac{1}{1 - \alpha}}} \Leftrightarrow \alpha > 1,$$

a contradiction. If \( \alpha < \eta \), then we similarly have

$$\theta^C < \left[ \frac{1 - \alpha}{(1 - \beta)\beta^{\frac{\alpha}{r}}} \right]^{\frac{1}{\frac{1}{1 - \alpha}}} \theta^B \quad \text{and} \quad \theta^B < \left\{ (1 - \beta) \beta^{\frac{\alpha}{r}} \left( \frac{P}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \alpha}} \right\}^{\frac{1}{\frac{1}{1 - \alpha}}} \Leftrightarrow \alpha > 1,$$

also a contradiction.

**Step 5.** We now prove part (ii) and suppose \( \theta^C < \theta^B \). We just need to consider the following case:

$$(1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{\alpha}{r}} \left( \theta^C \right)^{\frac{\alpha}{r}} < (1 - \beta) \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{\alpha}{r}} \left( \theta^B \right)^{\frac{\alpha}{r}},$$

which implies that

$$\left( \frac{\theta^B}{\theta^C} \right)^{\frac{\alpha}{r}} > \frac{1 - \alpha}{(1 - \beta)\beta^{\frac{\alpha}{r}}}.$$ \quad (45)

If \( \alpha > \eta \), then we have (44) as before. First, using \( \theta^C < \theta^B \), (30) and (44), we have

$$\theta^C < \left\{ (1 - \beta) \beta^{\frac{\alpha}{r}} \left( \frac{P}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \alpha}} \right\}^{\frac{1}{\frac{1}{1 - \alpha}}} \Leftrightarrow 1 - \alpha > \alpha(1 - \beta)\beta^{\frac{\alpha}{r}},$$

as desired. Second, using (30) and (44), we have

$$\theta^C < \left[ \frac{1 - \alpha}{(1 - \beta)\beta^{\frac{\alpha}{r}}} \right]^{\frac{1}{\frac{1}{1 - \alpha}}} \left\{ (1 - \beta) \beta^{\frac{\alpha}{r}} \left( \frac{P}{c} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{1 - \alpha}} \right\}^{\frac{1}{\frac{1}{1 - \alpha}}} \Leftrightarrow \alpha < 1,$$

as desired. If \( \alpha < \eta \), then we get from (45) that

$$1 > \frac{\theta^C}{\theta^B} > \left[ \frac{1 - \alpha}{(1 - \beta)\beta^{\frac{\alpha}{r}}} \right]^{\frac{1}{\frac{1}{1 - \alpha}}} \Leftrightarrow 1 - \alpha < (1 - \beta)\beta^{\frac{\alpha}{r}},$$

as desired. Also, note that (43) holds in this case. We hence get

$$\frac{\theta^C}{\theta^B} > \left[ \frac{1 - \alpha}{(1 - \beta)\beta^{\frac{\alpha}{r}}} \right]^{\frac{1}{\frac{1}{1 - \alpha}}} \Leftrightarrow 1 > \alpha,$$
as desired. Note that \( \alpha > \eta \iff 1 - \alpha < 1 - \eta \) and \( \alpha < \eta \iff 1 - \alpha > 1 - \eta \), thus the proof of part (ii) is complete.

**Step 6.** We now prove part (iii) and suppose \( \theta^C > \theta^B \). We just need to consider the following case:

\[
(1 - \alpha) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\alpha - \eta}} (\theta^C)_{\eta}^{\frac{1}{\alpha - \eta}} < (1 - \beta) \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\alpha - \eta}} (\theta^B)_{\eta}^{\frac{1}{\alpha - \eta}},
\]

for either \( \alpha > \eta \) or \( \alpha < \eta \). First, if \( \alpha > \eta \), then we get from this condition and (37) that

\[
1 > \frac{\theta^B}{\theta^C} > \left[ \frac{1 - \alpha}{(1 - \beta)\beta^{\frac{1}{\alpha - \eta}}} \right]^{\frac{\beta r - 1}{\beta - 1}} \quad \text{and} \quad \theta^B < \left\{ (1 - \beta)\beta^{\frac{1}{\alpha - \eta}} \left[ \frac{p}{c} \right]^{\frac{1}{\alpha - \eta}} \right\}^{\frac{1}{\beta - 1}} \left[ \frac{1 - \alpha}{(1 - \beta)\beta^{\frac{1}{\alpha - \eta}}} \right],
\]

by which we immediately obtain \( 1 - \alpha < \min \left\{ 1 - \eta, (1 - \beta)\beta^{\frac{1}{\alpha - \eta}} \right\} \) and also

\[
\left\{ (1 - \beta)\beta^{\frac{1}{\alpha - \eta}} \left[ \frac{p}{c} \right]^{\frac{1}{\alpha - \eta}} \right\}^{\frac{1}{\beta - 1}} \quad \text{by using (30) again, hence the desired assertion follows. Second, if } \alpha < \eta, \text{ then we similarly get}
\]

\[
\frac{\theta^C}{\theta^B} > \max \left\{ 1, \left[ \frac{1 - \alpha}{(1 - \beta)\beta^{\frac{1}{\alpha - \eta}}} \right]^{\frac{\beta r - 1}{\beta - 1}} \right\} \quad \text{and} \quad \theta^B > \left\{ (1 - \beta)\beta^{\frac{1}{\alpha - \eta}} \left[ \frac{p}{c} \right]^{\frac{1}{\alpha - \eta}} \right\}^{\frac{1}{\beta - 1}} \left[ \frac{1 - \alpha}{(1 - \beta)\beta^{\frac{1}{\alpha - \eta}}} \right],
\]

using which we are led to

\[
\theta^C > \left[ \frac{1 - \alpha}{(1 - \beta)\beta^{\frac{1}{\alpha - \eta}}} \right]^{\frac{1}{\alpha - \eta}} \left\{ (1 - \beta)\beta^{\frac{1}{\alpha - \eta}} \left[ \frac{p}{c} \right]^{\frac{1}{\alpha - \eta}} \right\}^{\frac{1}{\beta - 1}} \quad \text{\( \iff 1 > \alpha \)}
\]

and

\[
\theta^C > \left\{ (1 - \beta)\beta^{\frac{1}{\alpha - \eta}} \left[ \frac{p}{c} \right]^{\frac{1}{\alpha - \eta}} \right\}^{\frac{1}{\beta - 1}} \quad \text{\( \iff 1 - \alpha > \alpha(1 - \beta)\beta^{\frac{1}{\alpha - \eta}}, \)}
\]

in which we have used (30) again.

Finally, note that \( h^C = \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\alpha - \eta}} (\theta^C)_{\eta}^{\frac{1}{\alpha - \eta}}, h^B = \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\alpha - \eta}} (\theta^B)_{\eta}^{\frac{1}{\alpha - \eta}} \) and \( u^j = \delta / \left[ \delta + (\theta^j)^{1 - \eta} \right] \) for \( \forall j \in \{C, B\} \) under Assumption 4.1 and also the Hosios condition \( \eta = \beta \) is satisfied under Assumption 4.2, the desired assertion immediately follows. Q.E.D.

**Proof of Lemma 4.3:** We shall complete it in 4 steps.

**Step 1.** By using (30), (31) and (37), we have

\[
(1 - \beta) \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\alpha - \eta}} (\theta^B)_{\eta}^{\frac{1}{\alpha - \eta}} + (1 - \beta) \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\alpha - \eta}} (\theta^B)_{\eta}^{\frac{1}{\alpha - \eta}} - \beta c (\theta^B)_{\eta}^{\frac{1}{\alpha - \eta}} = (1 - \eta) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\alpha - \eta}} (\theta^B)_{\eta}^{\frac{1}{\alpha - \eta}} + (1 - \eta) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\alpha - \eta}} (\theta^B)_{\eta}^{\frac{1}{\alpha - \eta}} - \eta c (\theta^B)_{\eta}^{\frac{1}{\alpha - \eta}}.
\]
Step 2. We now prove part (i) by means of contradiction, and hence we assume that \( \theta^B = \theta^p \). Then it follows from (46) that

\[
0 = [(1 - \beta) - (1 - \eta)] c \left( \theta^p \right)^{1 - \eta} + \left(1 - \beta\right) \beta^{\frac{1}{r+\delta}} - (1 - \eta) \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} \left( \theta^p \right)^{\frac{1}{r+\delta}} + \left(1 - \beta\right) \beta^{\frac{1}{r+\delta}} - (1 - \eta) p \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} \left( \theta^p \right)^{\frac{1}{r+\delta} - \eta} \equiv \text{RHS}. \tag{47}
\]

It is easy to show that \( 1 - \beta > (1 - \beta) \beta^{\frac{1}{r+\delta}} > (1 - \beta) \beta^{\frac{1}{r+\delta}} \), hence \( 1 - \beta \leq 1 - \eta \) implies \( \text{RHS} < 0 \) and \( (1 - \beta) \beta^{\frac{1}{r+\delta}} \geq 1 - \eta \) implies \( \text{RHS} > 0 \), both violating (47). So, the desired assertion in part (i) follows.

Step 3. In what follows, we assume that \( \alpha = \eta \), which greatly simplifies (46) and leads us towards

\[
(1 - \beta) \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} + \left(1 - \beta\right) p \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} - \beta c \right) \left( \theta^p \right)^{1 - \eta} \equiv \text{RHS} \tag{48}
\]

It follows from Assumption 4.2 that \( 1 - \eta > (1 - \beta) \beta^{\frac{1}{r+\delta}} \), then we get from (48) that

\[
\left\{ (1 - \eta) p \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} - \eta c \right\} \left( \theta^p \right)^{1 - \eta} < \left\{ (1 - \beta) p \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} - \beta c \right\} \left( \theta^p \right)^{1 - \eta}. \tag{49}
\]

Here we consider three cases. First, if \( (1 - \beta) p \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} > \beta c \), then (49) implies that \( \theta^B > \theta^p \) for

\[
\left[ (1 - \eta) - (1 - \beta) \beta^{\frac{1}{r+\delta}} \right] p \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} \geq [(1 - \beta) - (1 - \eta)] c = 0,
\]

in which we have used the assumption \( 1 - \eta > (1 - \beta) \beta^{\frac{1}{r+\delta}} \) and Assumption 4.2. As a consequence, we get the corresponding result in part (ii). Second, if \( (1 - \beta) p \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} < \beta c \), then (49) implies that \( \theta^B < \theta^p \) for

\[
\left[ (1 - \eta) - (1 - \beta) \beta^{\frac{1}{r+\delta}} \right] p \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} \geq [(1 - \beta) - (1 - \eta)] c = 0,
\]

in which we have used the assumption \( 1 - \eta > (1 - \beta) \beta^{\frac{1}{r+\delta}} \) and Assumption 4.2. As a consequence, we get the corresponding result in part (ii). Third, if \( (1 - \beta) p \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} = \beta c \), then we have from (48) and (31) that \( (1 - \beta) \beta^{\frac{1}{r+\delta}} \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} = c(r + \delta) \), which however combined with the assumption \( (1 - \beta) p \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} = \beta c \) gives rise to \( \alpha = 1 \), a contradiction. Finally, note that

\[
\frac{p}{c} \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{r+\delta}} < \frac{\beta}{(1 - \beta) \beta^{\frac{1}{r+\delta}}} \iff p > \left[ \frac{\beta^{1 + \beta}(1 - \beta)^{1 - \beta}}{(r + \delta)c^{1 - \beta}} \right]^{\frac{1}{\beta}} \equiv \hat{p},
\]
\[ h^P = \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\alpha}} \left( \theta^P \right)^{\frac{\alpha(1-\eta)}{1-\eta}} \quad \text{and} \quad \theta^P = \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\alpha}} \left( \theta^B \right)^{\frac{\alpha(1-\eta)}{1-\eta}} \quad \text{for} \quad \forall j \in \{P, B\} \] under Assumption 4.1, the proof is therefore complete. Q.E.D.

**Proof of Lemma 4.4:** We shall complete it in 4 steps.

**Step 1.** Applying Assumption 4.1 to Lemmas 3.1-3.3 gives rise to:

\[ \mathcal{W}^j = f(h^j) - \left[ f(h^j) + ph^j + c\theta^j \right] u^j \]

\[ = \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\alpha}} \left( \theta^j \right)^{\frac{\alpha(1-\eta)}{1-\eta}} - \left\{ \left[ \frac{\alpha}{(r + \delta)p} \right]^{\frac{1}{\alpha}} \left( \theta^B \right)^{\frac{\alpha(1-\eta)}{1-\eta}} + p \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\alpha}} \left( \theta^B \right)^{\frac{\alpha(1-\eta)}{1-\eta}} \right\} \]

\[ \times \left[ \frac{\delta}{\theta^j + (\theta^B)^{1-\eta}} \right] = \tilde{\mathcal{W}}(\theta^j) \] (50)

for \( j \in \{C, P\} \), and

\[ \mathcal{W}^B = f(h^B) - \left[ f(h^B) + ph^B + c\theta^B \right] u^B \]

\[ = \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\alpha}} \left( \theta^B \right)^{\frac{\alpha(1-\eta)}{1-\eta}} - \left\{ \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\alpha}} \left( \theta^B \right)^{\frac{\alpha(1-\eta)}{1-\eta}} + p \left[ \frac{\alpha \beta}{(r + \delta)p} \right]^{\frac{1}{\alpha}} \left( \theta^B \right)^{\frac{\alpha(1-\eta)}{1-\eta}} \right\} \]

\[ \times \left[ \frac{\delta}{\theta^B + (\theta^B)^{1-\eta}} \right] = \tilde{\mathcal{W}}(\theta^B). \] (51)

**Step 2.** We first prove part (i). Given that we can write \( h \) and \( u \) as functions of \( \theta \), it follows from (50) that

\[ \frac{\partial \tilde{\mathcal{W}}}{\partial \theta} > 0 \iff cu \left[ 1 + \frac{\partial u}{\partial \theta} \left( \frac{\theta}{u} \right) \right] \left\{ f'(h) - u[f'(h) + p] \right\} \frac{\partial h}{\partial \theta} - [f(h) + ph] \frac{\partial u}{\partial \theta}. \]

By (9) and (13), we see that

\[ \varepsilon_{u, \theta} = \frac{\partial u}{\partial \theta} \left( \frac{\theta}{u} \right) = -\frac{\theta q(\theta)}{\delta + \theta q(\theta)} (1 - \eta) \in (-1, 0), \]

hence

\[ \frac{\partial \tilde{\mathcal{W}}}{\partial \theta} > 0 \iff c < \frac{\left\{ f'(h) - u[f'(h) + p] \right\} \frac{\partial h}{\partial \theta} - [f(h) + ph] \frac{\partial u}{\partial \theta}}{u \left[ 1 + \frac{\partial u}{\partial \theta} \left( \frac{\theta}{u} \right) \right]}. \]

Also, note that

\[ f'(h) > u[f'(h) + p] \iff u < \frac{(r + \delta) \theta^{-\eta - 1}}{(r + \delta) \theta^{-\eta - 1} + 1} \iff 0 < r \theta^{1-\eta}, \]

as desired in part (i).

**Step 3.** Even if we have established the strict monotonicity of the welfare function with respect to \( \theta \), mutual welfare comparison is immediate only only when the existence and uniqueness of equilibrium \( \theta \) are assured. It follows from Lemmas 3.1-3.3 that we should put \( \eta \geq 1/(2 - \alpha) \), which hence implies that \( \eta > \alpha \) (otherwise \( \alpha \geq \eta \Rightarrow (\alpha - 1)^2 \leq 0 \), an immediate contradiction).

By Step 2 we have

\[ \frac{\partial \tilde{\mathcal{W}}}{\partial \theta} = f'[h(\theta)] [1 - u(\theta)] h'(\theta) - \left\{ pu(\theta) h' + f[h(\theta)] u'(\theta) \right\} - ph(\theta) u'(\theta) - [u(\theta) + u'(\theta)\theta] c, \] (52)
in which \( h(\theta) = \left( \frac{\alpha}{(r + \delta)p} \right)^\frac{1}{\alpha} \theta^{1-\frac{\alpha}{r + \delta}} \) and \( u(\theta) = \delta / [\delta + \theta^{1-\eta}] \). First, it is easy to see that
\[
\lim_{\theta \downarrow 0} [u(\theta) + u'(\theta)\theta]c = c. \tag{53}
\]

Second, note that \( ph(\theta)u'(\theta) = -p \left[ \frac{\alpha}{(r + \delta)p} \right]^\frac{1}{\alpha} \left[ \frac{\delta(1-\eta)}{(r + \delta)^2 + \theta + \theta^{1-\eta}} \right] \theta^{1-2\eta + \alpha \eta} \) and \( 1 - 2\eta + \alpha \eta \leq 0 \Leftrightarrow \eta \geq 1/(2 - \alpha) \), thus we have
\[
-\infty < \lim_{\theta \downarrow 0} ph(\theta)u'(\theta) < 0 \quad \text{for} \quad \eta = 1/(2 - \alpha) \quad \text{and} \quad \lim_{\theta \downarrow 0} ph(\theta)u'(\theta) = -\infty \quad \text{for} \quad \eta > 1/(2 - \alpha). \tag{54}
\]

Third, note that
\[
- \left\{ pu(\theta)h'(\theta) + f[h(\theta)]u'(\theta) \right\} = (1-\eta) \left[ \frac{\alpha}{(r + \delta)p} \right] \left[ \frac{\alpha}{(r + \delta)p} \right]^\frac{1}{\alpha} \left[ \frac{1}{(r + \delta)(1-\eta)} \right] \left[ \frac{\delta}{\delta + \theta^{1-\eta}} \right] \theta^{(\alpha-\eta)/(1-\alpha)} \geq 0,
\]
thus we have
\[
\lim_{\theta \downarrow 0} \left( \left\{ pu(\theta)h'(\theta) + f[h(\theta)]u'(\theta) \right\} \right) = +\infty
\]
for either \( \alpha \leq 1/2 \) or \( r \geq (2\alpha - 1)\delta/(1 - \alpha) \). Finally, note that
\[
f'[h(\theta)][1 - u(\theta)]h'(\theta) = \left[ \frac{1-\eta}{1-\alpha} \right] \left[ \frac{\alpha}{(r + \delta)p} \right]^\frac{1}{\alpha} \left[ \frac{\delta}{\delta + \theta^{1-\eta}} \right] \theta^{(\alpha-\eta)/(1-\alpha)}
\]
and
\[
f'[h(\theta)][1 - u(\theta)]h'(\theta) - \left\{ pu(\theta)h'(\theta) + f[h(\theta)]u'(\theta) \right\}
\]
\[
= \left( \frac{1-\eta}{1-\alpha} \right) \left[ \frac{\alpha}{(r + \delta)p} \right]^\frac{1}{\alpha} \left[ \frac{(1-\alpha)\delta^2 + r(\delta + \alpha\theta^{1-\eta})}{(r + \delta)(\delta + \theta^{1-\eta})} \right] \left( \frac{1}{\delta + \theta^{1-\eta}} \right) \theta^{(\alpha-\eta)/(1-\alpha)} > 0,
\]
thus we actually always have
\[
\lim_{\theta \downarrow 0} f'[h(\theta)][1 - u(\theta)]h'(\theta) - \left\{ pu(\theta)h'(\theta) + f[h(\theta)]u'(\theta) \right\} = +\infty. \tag{55}
\]
As a consequence, by applying equations (53)-(55), we get from equation (52) that
\[
\lim_{\theta \downarrow 0} \frac{\partial \tilde{V}}{\partial \theta} > 0
\]
for any \( c < +\infty \). Since it is easy to show that \( \partial \tilde{V}/\partial \theta \) is continuous with respect to \( \theta \), we thus can find a critical value of \( \theta \) which is strictly positive such that \( \partial \tilde{V}(\theta)/\partial \theta > 0 \) always holds true for any \( \theta \) smaller than or equalling to this critical value and also any \( c < +\infty \). This hence completes the proof of part (i).

**Step 4.** For any given \( \theta \in (0, \infty) \), we get from (50)-(51) that
\[
\tilde{W}(\theta) - \hat{W}(\theta) = \left( 1 - \beta \frac{\alpha}{(r + \delta)p} \right) \left( \frac{\alpha}{r + \delta} \right)^\frac{1}{\alpha} \left( \frac{\theta^{1-\eta}}{\delta + \theta^{1-\eta}} \right) \left( \frac{\delta}{\delta + \theta^{1-\eta}} \right) p^{\frac{1}{\alpha}},
\]
simplifying which produces the required assertion in part (ii). Q.E.D.